# Transitions from deterministic to stochastic diffusion

R. Klages*
*Max Planck Institute for Physics of Complex Systems, Nöthnitzer Str. 38, D-01187 Dresden, Germany*
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We examine characteristic properties of deterministic and stochastic diffusion in low-dimensional chaotic dynamical systems. As an example, we consider a periodic array of scatterers defined by a simple chaotic map on the line. Adding different types of time-dependent noise to this model we compute the diffusion coefficient from simulations. We find that there is a crossover from deterministic to stochastic diffusion under variation of the perturbation strength related to different asymptotic laws for the diffusion coefficient. Typical signatures of this scenario are suppression and enhancement of normal diffusion. Our results are explained by a simple theoretical approximation.

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To understand diffusion in *noisy maps*, that is, in time-discrete dynamical systems where the deterministic equations of motion are perturbed by noise, figures as a prominent problem in recent literature. The most simple example of such models are one-dimensional chaotic maps on the line. In seminal contributions by Geisel and Nierwetberg [1], and by Reimann et al. [2], scaling laws have been derived for the diffusion coefficient yielding suppression and enhancement of diffusion with respect to variation of the noise strength. Related results have been obtained in Refs. [3,4]. However, all these results apply only to the onset of diffusion where the scaling laws are reminiscent of a dynamical phase transition, and not much appears to be known far away from this transition point. In such more general situations, only perturbations by a nonzero average bias have been studied [5]. Related models are deterministic Langevin equations, in which the interplay between deterministic and stochastic chaos has been analyzed [6], however, without focusing on diffusion coefficients. Non-diffusive noisy maps have furthermore been investigated by refinements of cycle expansion methods [7].

In this work we study the transition scenario from deterministic to stochastic diffusion in the most simple type of chaotic dynamical systems, which are piecewise linear maps on the line. Particularly, we are searching for signatures of deterministic and stochastic dynamics in the diffusion coefficient as a function of the strength of time-dependent stochastic noise. In this aspect our work appears to be related to the recent dispute on a possible distinction between chaotic and stochastic diffusion in experiments [8], where some of the theoretical models studied are very similar to the one introduced below.

We define our system as follows: The unperturbed map is given by the equation of motion

$$x_{n+1} = M_a(x_n) \quad , \tag{1}$$

where $a \in \mathbb{R}$ is a control parameter and $x_n$ is the position of a point particle at discrete time $n$. $M_a(x)$ is continued periodically beyond the interval $[-1/2, 1/2)$ onto the real line by a lift of degree one, $M_a(x+1) = M_a(x) + 1$. We assume that $M_a(x)$ is anti-symmetric with respect to $x = 0$, $M_a(x) = -M_a(-x)$. The map we study as an example is defined by $M_a(x) = ax$, where the uniform slope $a$ serves as a control parameter. The Lyapunov exponent of this map is given by $\lambda = \ln a$ implying that for $a > 1$ the dynamics is chaotic. We now apply two types of annealed disorder to this map, (i) noisy slopes [2,4]: we add the random variable $\Delta a_n$, $n \in \mathbb{N}$, to all slopes $a$ making them time-dependent in form of

$$M_{a+\Delta a_n}(x) = (a + \Delta a_n)x \ , \tag{2}$$

or (ii) noisy shifts [1–3]: we add the random variable $\Delta b_n$, $n \in \mathbb{N}$, as a time-dependent uniform bias yielding

$$M_{a,\Delta b}(x) = ax + \Delta b_n \ . \tag{3}$$

In both cases we assume that the random variable $\Delta_n \in \{\Delta a_n, \Delta b_n\}$ is independent and identically distributed according to a distribution $\chi_d(\Delta_n)$, where $d \in \{da, db\}$ is again a control parameter. In the following we will consider two different types of such distributions, namely random variables distributed uniformly over an interval of size $[-d, d]$ [3,4],

$$\chi_d(\Delta_n) = \frac{1}{2d}\Theta(d + \Delta_n)\Theta(d - \Delta_n) \quad , \tag{4}$$

and dichotomous or $\delta$-distributed random variables [2,4],

$$\chi_d(\Delta_n) = \frac{1}{2}(\delta(d-\Delta_n)+\delta(d+\Delta_n)) \quad . \tag{5}$$

Since $|\Delta_n| \le d$, we denote $d$ as the perturbation strength. As an example, we sketch in Fig. 1 our model for noisy slopes. We now define the diffusion coefficient as

$$D(a,d) = \lim_{n\to\infty} \frac{1}{2n}(< x_n^2 >_{\rho_0} - < x_n >_{\rho_0}^2) \ , \tag{6}$$

with

$$< x_n^k >_{\rho_0} = \int dx \int d(\Delta_0)d(\Delta_1)\ldots d(\Delta_{n-1})\rho_0(x)\chi(\Delta_0)\chi(\Delta_1)\ldots\chi(\Delta_{n-1})x_n^k \ , \tag{7}$$

where $\rho_0(x)$ denotes the initial distribution of an ensemble of moving particles, $x_0 \equiv x$, $k \in \mathbb{N}$, and $\Delta_j, j \in \{1,\ldots,n-1\}$, is the random variable. Note that in computer simulations it suffices to generate a single series of random variables instead of evaluating all the integrals in Eq. (7). To obtain better numerical convergence for noisy shifts the current squared in Eq. (6) was subtracted at any time step while Eqs. (4),(5) imply that the long-time average over the random variable $\Delta_n$ does not yield any bias. In Refs. [10,11] it was shown that the unperturbed map Eq. (1) exhibits normal diffusion if $a > 2$, and the same was found recently by adding a bias $b$ [12]. Correspondingly, for the types of perturbations defined above diffusion should always be normal if $(a - da) > 2$, as was confirmed in simulations. Hence, the central question is what happens to the parameter-dependent diffusion coefficient $D(a,d)$ under variation of the two control parameters $a$ and $d$ in case of the above two types of noise.

For $da = 0$ it was shown that the unperturbed diffusion coefficient $D(a,0)$ is a fractal function of the slope $a$ as a control parameter [10,11], as is depicted again in Fig. 1. Included are results from computer simulations for uniformly distributed noisy slopes at different values of the perturbation strength $da$ [13]. As expected, the fractal structure gradually smoothes out by increasing $da$. Qualitatively the same result is obtained by applying noisy shifts [14]. Fig. 1 may be compared to the corresponding result for *quenched* slopes Fig. 1 in Ref. [9]. Apart from numerical uncertainties, there are clear differences in the critical behavior close to the onset of diffusion. However, for small enough perturbation strength and large enough $a$ the results look qualitatively similar indicating that in this limit quenched and annealed diffusion may be treated on the same footing.

More evidence for this statement is obtained from a trivial approximation for the perturbed diffusion coefficient, which we motivate starting from dichotomous noisy slopes. Naive reasoning suggests that, at arbitrary fixed parameters $a$ and $da$, the *perturbed* diffusion coefficient $D(a,da)$ can be approximated by simply averaging over the *unperturbed* diffusion coefficient $D(a,0)$ at respective values of the slopes $a - da$ and $a + da$ yielding $D_{\rm app}(a,da) = (D(a-da,0)+D(a+da,0))/2$. This heuristic argument can be straightforwardly extended to the random distribution Eq. (4) as well as to any other type of uncorrelated noise yielding the generalized expression

$$D_{\rm app}(\boldsymbol{p},\boldsymbol{d}) = \int d(\boldsymbol{\Delta})\ \chi_d(\boldsymbol{\Delta})D(\boldsymbol{p}+\boldsymbol{\Delta},0) \ . \tag{8}$$

Here $\boldsymbol{p}$ is a vector of control parameters such as $\boldsymbol{p} = \{a,b\}$ in case of the map above, $\boldsymbol{d}$ is the corresponding vector of perturbation strengths, and $\boldsymbol{\Delta}$ is the vector of perturbations such as $\boldsymbol{\Delta} = \{\Delta a, \Delta b\}$ for noisy shifts and slopes. Further generalizations of this equation, for example, to arbitrary moments as defined in Eq. (7), are straightforward. Applying this formula to the case of quenched slopes discussed in Ref. [9] reproduces the diffusion coefficient approximation Eq. (6) therein, which was obtained in the limit of small perturbation strength. The corresponding approximations for uniform noisy slopes are depicted in Fig. 1 as lines. They show that even for the rather large perturbation strength $da = 1$ the agreement between theory and simulations is excellent. This confirms that, in the limit described above, quenched and annealed disorder generating normal diffusion can indeed approximately be treated in the same way.

Let us now look at the diffusion coefficient for a given value of $a$ as a function of $da$. Fig. 1 shows that approximately at odd and even integer slopes the fractal diffusion coefficient $D(a,0)$ exhibits a local maximum or minimum, respectively. Since Eq. (8) represents an average over the unperturbed solution in a local environment $[a-da, a+da]$ it predicts local suppression and enhancement of diffusion at odd and even integer slopes, respectively, under variation of the perturbation strength $da$. This has already been conjectured in Ref. [11] and has been verified in Ref. [9] for quenched slopes. We first check this hypothesis for noisy slopes around the local maximum of $D(a,0)$ at $a = 7$ distributed according to Eqs. (4),(5). Figs. 2 (a), (b) depict again results obtained from computer simulations in comparison to Eq. (8). As predicted, in both cases there is suppression of diffusion for small enough $da$. For dichotomous noise the perturbed diffusion coefficient increases on a coarse scale by exhibiting multiple, fractal-like suppression and enhancement on finer scales. For uniform perturbations there is a pronounced crossover from suppression to enhancement on a coarse scale, by again exhibiting oscillations on a fine scale. In both cases the agreement between

simple theory and simulations is excellent for small enough $da$, whereas clear systematic deviations particularly in case of dichotomous noise are visible for larger $da$. Note that if $a - \Delta a_n < 2$ particles are getting trapped within a box at a respective time step $n$, and that for $a - \Delta a_n < 1$ the map is non-chaotic. In the first case simulations and simple reasoning suggest that the perturbed map still exhibits normal diffusion. However, as soon as $a - da < 1$ numerical results indicate that there is no normal diffusion anymore [14]. This appears to be due to the contracting behavior of the non-chaotic map resulting in localization of particles. The oscillatory behavior of the diffusion coefficient in Fig. 2 (a) just below this transition point is not yet understood.

Employing Eq. (8) we now analyze noisy shifts. The unperturbed two-parameter diffusion coefficient $D(a, b, 0)$ has been calculated numerically exactly for the map under consideration in Ref. [12]. Results for the perturbed diffusion coefficient $D(a, db) \equiv D(a, 0, db)$ are presented in Fig. 3 (a) for dichotomous noise and (b) for uniform perturbations, both starting from $D(a, 0)$ at $a = 6$. In both cases the perturbed diffusion coefficient exhibits strong enhancement of diffusion for small enough perturbation strength due to the fact that the unperturbed diffusion coefficient at $a = 6$ is approximately identical with a local maximum in the $(a, b)$ parameter plane [12]. For dichotmous perturbations it suffices to show results for $0 < db < 0.5$ only. Translation and reflection symmetry of the map imply that this function is mirrored in the interval from $0.5 < db < 1$, and that the full sequence in $0 < db < 1$ is periodically repeated for higher values of $db$. As in the corresponding case of noisy slopes, the perturbed diffusion coefficient increases on a coarse scale by exhibiting multiple fractal-like suppression and enhancement on a fine scale. In case of uniform perturbations there is a pronounced crossover to an approximately constant diffusion coefficient for larger $db$.

Before calculating the stochastic limit of the diffusion coefficient we provide a simple analytical justification for the heuristic approximation Eq. (8). For sake of simplicity, we demonstrate it only for noisy slopes, $\Delta_n \equiv \Delta a_n$. Noisy shifts as well as quenched disorder can be treated along the same lines [14]. Let us start from the definition of the diffusion coefficient Eq. (6) where $< x_n >= 0$. Let $\Delta a_n$ be uniformly distributed in $[-da, da]$, $\Delta a_0 \equiv \Delta a$. In case of $da \to 0$ all random variables are bounded by $\Delta a_n = \Delta a + \epsilon$, $-2da \le \epsilon \le 2da$. We now put this expression into the perturbed equation of motion Eqs. (1),(2), as contained in Eq. (6), which we write as $x_{n+1,a+\Delta a_n} = M_{a+\Delta a_n}(x_n)$. As a first step we now take the limit $\epsilon \to 0$ resulting in the expression for the mean square displacement

$$\begin{aligned} < x_n^2 > &= \int dx \int d(\Delta a) d(\Delta a_1) \dots d(\Delta a_{n-1}) \rho_0(x) \chi(\Delta a) \chi(\Delta a_1) \dots \chi(\Delta a_{n-1}) x_{n,a+\Delta a_{n-1}}^2 \\ &= \int dx \int d(\Delta a) \rho_0(x) \chi(\Delta a) x_{n,a+\Delta a}^2 \ (\epsilon \to 0) \ . \end{aligned} \tag{9}$$

As a second step we exchange the time limit contained in Eq. (6) with the integration over $d(\Delta a)$ yielding

$$\begin{aligned} D_{app}(a, da) &= \lim_{n\to\infty} \frac{< x_n^2 >}{2n} \\ &= \int d(\Delta a) \chi_{da}(\Delta a) \lim_{n\to\infty} \int dx \rho_0(x) \frac{x_{n,a+\Delta a}^2}{2n} \\ &= \int d(\Delta a) \chi_{da}(\Delta a) D(a + \Delta a, 0) \ , \end{aligned} \tag{10}$$

where we have used that the unperturbed diffusion coefficient was defined as

$$D(a, 0) = \lim_{n\to\infty} \int dx \rho_0(x) x_{n,a}^2 \ . \tag{11}$$

We have thus verified our previous approximation Eq. (8) for noisy slopes in the limit of small perturbation strength. A similar derivation can be carried out for noisy shifts arriving again at Eq. (8) in case of very small perturbation strength. For quenched shifts it is known that a normal diffusion coefficient does not exist [15], thus any approximation by Eq. (8) must fail. Indeed, it turns out that in this case taking the limit $\epsilon \to 0$ fundamentally changes the properties of the dynamical system and is thus no valid operation [14].

Finally, we calculate the parameter-dependent stochastic diffusion coefficient related to the map with noisy slopes. Starting from the definition Eq. (11) the complete loss of memory in the unperturbed map is modeled by [11,16] (i) replacing the distance $x_n$ a particle travels by $n$ times the distance a particle travels at any single time step, $n\Delta x = n(M_a(x) - x)$, and (ii) neglecting any memory effects in the probability density on the unit interval by assuming $\rho_0(x) = 1$. Then Eq. (11) yields

$$D_{rw}(a) = \frac{(a-1)^2}{24} \ . \tag{12}$$

As was shown in Refs. [11,16], this equation correctly describes the asymptotic parameter dependence of the deterministic diffusion coefficient for $a \to \infty$ thus explaining the increase of $D(a, 0)$ in Fig. 1 on a coarse scale. On this basis, the corresponding result for noisy slopes is easily calculated by using Eq. (12) as the functional form for $D(a + \Delta a, 0)$ in the approximation Eq. (10) reading

$$D_{rw}(a, da) = D_{rw}(a, 0) + \Delta a^2/c \; , \tag{13}$$

where $c = 24$ for dichotomous noise Eq. (5) and $c = 72$ for uniform noise Eq. (4). Eq. (13) thus confirms the common sense expectation that noise should typically enhance diffusion and represents the *stochastic limit* of the diffusion coefficient. This equation is depicted in Fig. 2 (a), (b) in form of dashed lines. In case of dichotomous noise the correlations are apparently large enough such that even for large perturbation strength $da$ there is no transition to the stochastic limit, whereas in case of uniform noisy slopes the diffusion coefficient approaches the stochastic solution asymptotically in $da$ thus verifying the existence of a transition from deterministic to stochastic diffusion. That such a distinct transition behavior exists in these models was already conjectured in Ref. [11]. Analogous calculations for noisy shifts yield Eq. (12) for all values of $db$ reflecting the fact that for large enough $a$ the stochastic diffusion coefficient should not depend on the bias. This result is shown in Fig. 3 (b) and again confirms an asymptotic approach of the diffusion coefficient to the stochastic limit under variation of $db$. Based on the known result of the existence of a fractal diffusion coefficient for the unperturbed $D(a, b, 0)$ we conjecture that the typical transition scenario in this type of systems consists of (multiple) suppression and enhancement of diffusion. We finally note that Eqs. (12),(13) are closely related to the approximation outlined in Ref. [1], and to the simple heuristic argument given by Reimann [2] by which he explains the suppression of deterministic diffusion by noise in the climbing sine map near a crisis; more details will be discussed elsewhere [14].

We conclude with a few remarks: (1) It would be interesting to study the problem of noisy maps with non-zero average bias along the same lines. Ref. [12] shows that the unperturbed map does not exhibit linear response for $b \to 0$, thus we conjecture that adding noise generates a transition to Ohm's law. (2) In the recent Ref. [17] a rather general mechanism of noise suppression by noise has been reported. Whether there is a more detailed relation between the argument outlined in this reference and the phenomena discussed here appears to be an open question. (3) Our approach may be useful to investigate the impact of noise on the diffusion coefficient in more complex time-continuous systems as well. In particular, we are thinking of models such as the standard map, particle billiards, or inertia ratchets, where irregular transport coefficients have already been reported and studied under the impact of noise [18]. However, these analyses were not performed from the point of view of suppression and enhancement of diffusion, or by looking for transitions to the stochastic limit.

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* Electronic address: rklages@mpipks-dresden.mpg.de

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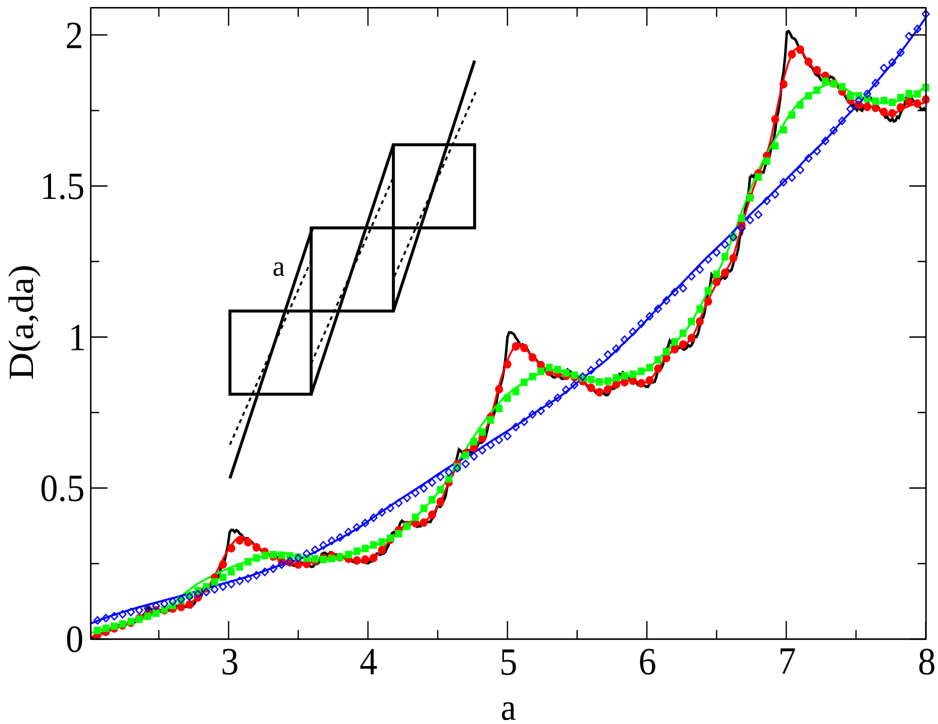


FIG. 1. Diffusion coefficient $D(a, da)$ for the piecewise linear map shown in the figure. The slope $a$ is perturbed by uniform noise of maximum strength $da$ as defined in Eq. (4). The bold black line depicts numerically exact results for the unperturbed diffusion coefficient at $da = 0$. Computer simulation results for $da \neq 0$ are marked with symbols, the corresponding lines are obtained from the approximation Eq. (8). The parameter values are: $da = 0.1$ (circles), $da = 0.4$ (squares), $da = 1.0$ (diamonds).

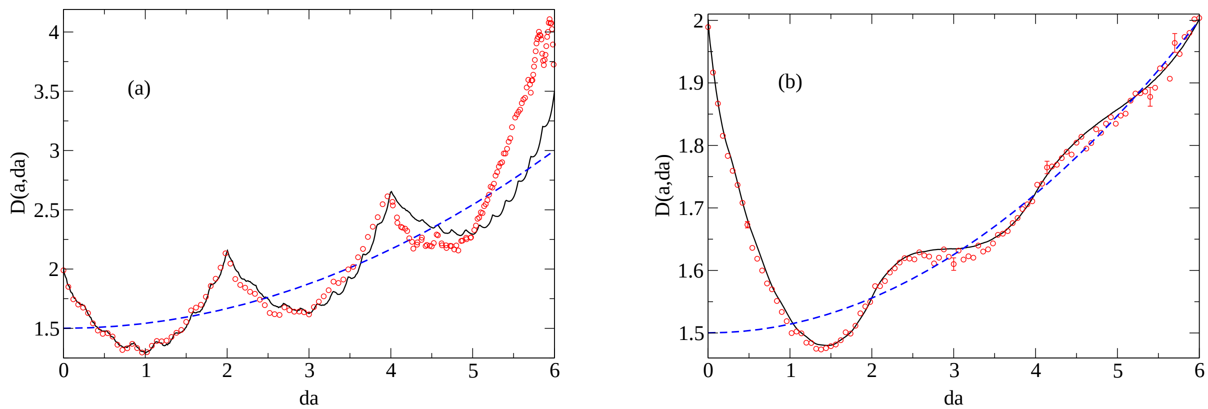


FIG. 2. Diffusion coefficient $D(a, da)$ as a function of the perturbation strength $da$ at slope $a = 7$ for noisy slopes distributed according to: (a) dichotomous noise Eq. (5), (b) uniform noise Eq. (4). The circles represent results from computer simulations, the bold lines are obtained from the approximation Eq. (8), the dashed lines represent the stochastic limit for the diffusion coefficient Eq. (13).

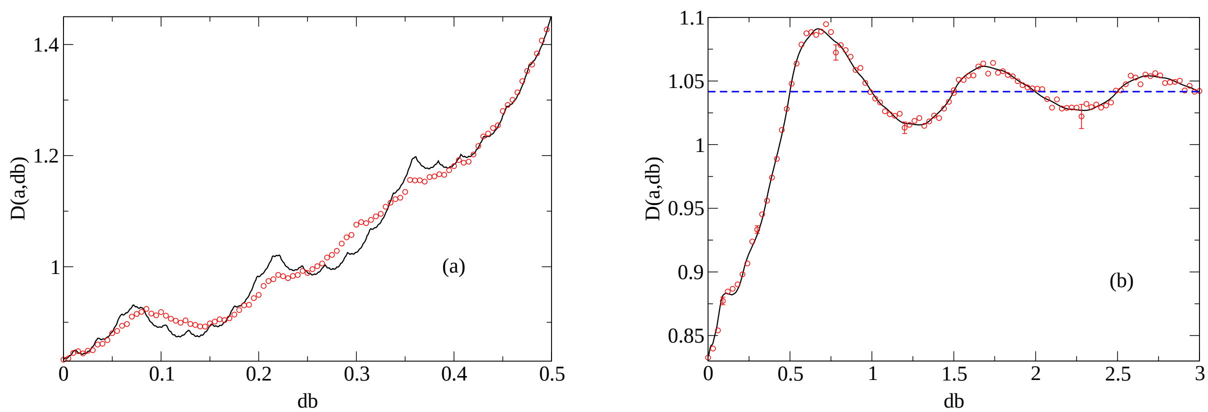


FIG. 3. Diffusion coefficient $D(a, db)$ as a function of the perturbation strength $db$ at slope $a = 6$ for noisy shifts distributed according to: (a) dichotomous noise Eq. (5), (b) uniform noise Eq. (4). The circles represent results from computer simulations, the bold lines are obtained from the approximation Eq. (8), the dashed line represents the stochastic limit for the diffusion coefficient Eq. (12).